ERRATUM TO MOTIVIC HOMOLOGICAL STABILITY FOR CONFIGURATION SPACES OF THE LINE

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Abstract. The purpose of this note is to explain an error in the paper [Hor16] and to give a correct proof of a weaker version (see Theorem 2.3) of the main theorem in that paper. I am grateful to Tom Bachmann for spotting this mistake.

1. Explanation of the error

It is claimed in [Hor16, Proposition 4.5] that the Betti realization functor $B : \mathbf{DMT}(\mathbf{Z}) \to \mathbf{D}(\mathbf{Z})$ is conservative. This is incorrect. Indeed, the number -1 is a non-trivial 2-root of unity defined over $\mathbf{Z}$, hence it is classified by a non-zero map $\tau : \mathbf{Z}/2(0) \to \mathbf{Z}/2(1)$ in $\mathbf{DMT}(\mathbf{Z})$. The map τ is not an isomorphism in $\mathbf{DMT}(\mathbf{Z})$ but the Betti realization functor sends τ to the identity map of $\mathbf{Z}/2$.

Let us also mention that there are analogues of the maps τ at each prime. This failure of conservativity ruins the proof of the main theorem of [Hor16] on motivic cohomological stability for configuration spaces of the affine line. However, it could still be that this statement is true and provable by another method.

2. Correction for étale motives

In this section we prove that homological stability for configurations spaces of points in the affine line holds for étale motivic cohomology. For Λ a commutative ring, and S any base scheme we denote by $\mathbf{DA}(S, \Lambda)$ the triangulated category of étale motives over $\mathrm{Spec}(\mathbf{Z})$ with coefficients in Λ defined in [?]. This is obtained from the category of complexes of étale sheaves on smooth schemes over S by forcing $\mathbb{A}^1$-invariance and invertibility of the Tate twist. When $\Lambda = \mathbf{Z}$, we simply write $\mathbf{DA}(\mathbf{Z})$. We denote by $\mathbf{DAT}(\mathbf{Z}, \Lambda)$ the smallest triangulated category of $\mathbf{DA}(\mathbf{Z}, \Lambda)$ containing the Tate twists $\Lambda(n)$ for $n \in \mathbf{Z}$. For $S = \mathrm{Spec}(R)$ with R a subring of $\mathbf{C}$, there is a Betti realization functor $B : \mathbf{DA}(S, \Lambda) \to \mathbf{D}(\Lambda)$ that sends the motive of a smooth $\mathbf{Z}$-scheme X to $C_*^{sing}((X \otimes_R \mathbf{C})_{an}, \Lambda)$ (see e.g [?, Section 2.1] for a construction). There is also an étale realization functor

$$E : \mathbf{DA}(\mathbf{Z}[1/p]) \to \mathbf{D}(\mathrm{Spec}(\mathbf{Z}[1/p])_{et}, \mathbf{Z}/p^k)$$

We now recall the construction of this functor following [?]. For any base scheme S, there is an inclusion of sites $S_{et} \to \mathrm{Sm}_S$ from étale schemes over S to smooth schemes over S, this induces a functor

$$\mathbf{D}(S_{et}, \mathbf{Z}/p^k) \to \mathbf{D}(\mathrm{Sm}_S, \mathbf{Z}/p^k)$$

between the derived categories of étale sheaves on both sides. Postcomposing with the functor $\mathbf{D}(\mathrm{Sm}_S, \mathbf{Z}/p^k) \to \mathbf{DA}(S, \mathbf{Z}/p^k)$ we obtain a left adjoint

$$\mathbf{D}(S_{et}, \mathbf{Z}/p^k) \to \mathbf{DA}(S, \mathbf{Z}/p^k)$$

This left adjoint is an equivalence in good cases (see [**?**, Théorème 4.1] for the precise hypothesis). In those cases, the right adjoint is thus also a left adjoint and is the definition of E.

We recall Artin's result comparing étale and Betti realization.

Theorem 2.1. *Let $S = \mathrm{Spec}_{\mathbf{C}}$, then the functors B and E from $\mathbf{DA}(S, \mathbf{Z}/p^k)$ to $\mathbf{D}(\mathbf{Z}/p^k)$ are naturally equivalent.*

We can now prove the main technical result.

Proposition 2.2. *Let M be an object of $\mathbf{DAT}(\mathbf{Z})$ that is such that $B(M)$ is in $\mathbf{D}(\mathbf{Z})_{\geq 0}$. Then for all integers n and all positive integers i, we have*

$$\mathrm{Hom}_{\mathbf{DAT}(\mathbf{Z})}(M, \mathbf{Z}(n)[-i]) = 0$$

Proof. Let p^k be a prime power. We write N for $M \otimes^{\mathbb{L}} \mathbf{Z}/p^k$. We observe first that $\mathrm{Hom}_{\mathbf{DAT}(\mathbf{Z})_{\mathbf{Z}/p^k}}(N, \mathbf{Z}/p^k(n)[-i]) = 0$ for any n and any $i > 0$. To prove this let us consider the following commutative square of left adjoint functors

$$\begin{tikzcd}
\mathbf{DA}(\mathbf{Z}, \mathbf{Z}/p^k) \arrow[r, "p^*"] & \mathbf{DA}(\mathbf{Z}[1/p], \mathbf{Z}/p^k) \arrow[r, "E"] \arrow[d, "q^*"'] & \mathbf{D}(\mathrm{Spec}(\mathbf{Z}[1/p])_{et}, \mathbf{Z}/p^k) \arrow[d, "q^*"] \\
& \mathbf{DA}(\mathbf{C}, \mathbf{Z}/p^k) \arrow[r, "E"'] & \mathbf{D}(\mathbf{Z}/p^k)
\end{tikzcd}$$

where the two horizontal maps in the square are étale realization functors, the two vertical maps are the functors induced by the map $q : \mathrm{Spec}\,\mathbf{C} \to \mathrm{Spec}(\mathbf{Z}[1/p])$ and the left horizontal map is induced by the map $p : \mathrm{Spec}(\mathbf{Z}[1/p]) \to \mathrm{Spec}(\mathbf{Z})$. All the horizontal maps in this diagram are equivalences of categories (for p^* this follows from [**?**, Proposition A.3.4]). Moreover, the triangulated category $\mathbf{D}(\mathrm{Spec}(\mathbf{Z}[1/p])_{et}, \mathbf{Z}/p^k)$ is the derived category of an abelian category and as such it has a t-structure in which an object is connective (resp. coconnective) if its cohomology sheaves are zero in negative (resp. non-negative degree).

We claim that the functor q^* reflects connective and coconnective objects (i.e. an object is connective or coconnective if and only if its image by q^* is so with respect to the standard t-structure on $\mathbf{D}(\mathbf{Z}/p^k)$). Let us first recall a few facts about the category of étale sheaves on $\mathbf{Z}[1/p]$. By [**?**, I. Corollaire 10.3], there is an equivalence of categories between étale covers of $\mathrm{Spec}(\mathbf{Z}[1/p])$ and finite extensions of $\mathbf{Q}$ that are unramified away from p (recall that an extension L of $\mathbf{Q}$ is unramified at a prime ℓ if $\mathcal{O}_L/\ell$ has no nilpotent elements, where $\mathcal{O}_L$ is the ring of integers in L). We can then consider the field K which is the union of all the subfields of $\overline{\mathbf{Q}}$ that are unramified away from p. The Galois group Γ of this field over $\mathbf{Q}$ is a profinite group and we can identify the category of sheaves of sets on $(\mathrm{Spec}(\mathbf{Z}[1/p]))_{et}$ with the category of sets with a continuous Γ-action. Likewise, the category of sheaves of $\mathbf{Z}/p^k$-modules on this site is equivalent to the category of $\mathbf{Z}/p^k$-modules equipped with a continuous Γ-action. Seen through this equivalence, the functor q^* is just the functor that sends a complex with Γ-action to the underlying complex. From this description, the claim that q^* reflects connective and coconnective objects is obvious.

Since Ep^* is an equivalence of categories, we have an isomorphism

$$\mathrm{Hom}_{\mathbf{DAT}(\mathbf{Z},\mathbf{Z}/p^k)}(N, \mathbf{Z}/p^k(n)[-i]) \cong \mathrm{Hom}_{\mathbf{D}(\mathrm{Spec}(\mathbf{Z}[1/p])_{et}, \mathbf{Z}/p^k)}(Ep^*N, Ep^*(\mathbf{Z}/p^k(n))[-i])$$

Since the Betti realization of M is connective, it is also the case for the Betti realization of N and and thus by the previous theorem, the fact that $Eq^* \cong q^*E$ and the fact that q^* reflects connectivity, we deduce that Ep^*N is connective. The same argument shows that

$Ep^*(\mathbf{Z}/p^k(n))[-i]$ is coconnective. Hence, we have

$$\mathrm{Hom}_{\mathbf{DAT}(\mathbf{Z},\mathbf{Z}/p^k)}(N,\mathbf{Z}/p^k(n)[-i])=0$$

Since the functor $-\otimes^{\mathbb{L}}\mathbf{Z}/p^k:\mathbf{DAT}(\mathbf{Z})\to\mathbf{DAT}(\mathbf{Z},\mathbf{Z}/p^k)$ is left adjoint to the obvious forgetful functor, we also have

$$\mathrm{Hom}_{\mathbf{DAT}(\mathbf{Z})}(M,\mathbf{Z}/p^k(n)[-i])=0$$

The long exact sequence associated to the triangle $\mathbf{Z}(n)\to\mathbf{Z}(n)\to\mathbf{Z}/p^k(n)$ shows that multiplication by p^k is an isomorphism of the abelian group $\mathrm{Hom}_{\mathbf{DAT}(\mathbf{Z})}(M,\mathbf{Z}(n)[-i])$. It follows that this group is a rational vector space and we are thus reduced to proving that

$$\mathrm{Hom}_{\mathbf{DAT}(\mathbf{Z},\mathbf{Q})}(M\otimes\mathbf{Q},\mathbf{Q}(n)[-i])=0$$

But this follows from the fact that $\mathbf{DAT}(\mathbf{Z},\mathbf{Q})$ has a t-structure for which an object is connective (resp. coconnective) if and only if its Betti realization is connective (resp. coconnective). Indeed, we have an equivalence of categories $\mathbf{DAT}(\mathbf{Z},\mathbf{Q})\simeq\mathbf{DMT}(\mathbf{Z},\mathbf{Q})$ and the proof of [Hor16, Corollary 4.6] is correct with rational coefficients. □

We can now state the homological stability theorem for étale motives. For X a smooth scheme over $\mathbf{Z}$, we denote by $\mathrm{H}^i_{et}(X,\mathbf{Z}(q))$ the group of homomorphism from the motive of X to $\mathbf{Z}(q)[i]$ in the triangulated category $\mathbf{DA}(\mathbf{Z})$.

Theorem 2.3. *There is an isomorphism*

$$\mathrm{H}^i_{et}(C_d,\mathbf{Z}(q))\cong\mathrm{H}^i_{et}(C_{d+1},\mathbf{Z}(q))$$

for any q and for $i<l(d)=\min(d,\lfloor d/2\rfloor+2)$

Proof. As in [Hor16], this isomorphism is induced by a zig-zag

$$C_d\to F_d\xrightarrow{\alpha}F_{d+1}\leftarrow C_{d+1}$$

where the two extreme maps are scanning maps and the middle map is the map α from the paragraph following Lemma 6.2 in [Hor16]. We refer the reader to section 5 of [Hor16] for the definition of the schemes C_d and F_d and the scanning map $C_d\to F_d$.

It suffices to show that each of these three maps induce an isomorphism on $\mathrm{H}^i_{et}(-,\mathbf{Z}(q))$ in the given range. According to Proposition 2.2, it is enough to show that the Betti realization of each of these three maps has a fiber that is at least $l(d)$-connected. For α, this is done in the proof of [Hor16, Proposition 6.4] and for the other two maps scanning maps this is done in the proof of [Hor16, Theorem 6.5]. □

Finally, we remark that the analogue of [Hor16, Proposition 4.7] is true for étale motivic cohomology. More precisely, we can consider the ind-scheme F_∞ and associate to it a motive in $\mathbf{DA}(\mathbf{Z})$ by taking the colimit of the motives of F_d. We then have the following.

Proposition 2.4. *For $i<l(d)$, we have an isomorphism*

$$\mathrm{H}^i_{et}(C_d,\mathbf{Z}(q))\cong\mathrm{H}^i_{et}(F_\infty,\mathbf{Z}(q))$$

References

[Hor16] G. Horel. Motivic homological stability for configuration spaces of the line. *Bull. Lond. Math. Soc.*, 48(4):601–616, 2016.

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